Path integrals and geometry of trajectories

Matthias Blau [1]
Centre de Physique Théorique II, Laboratoire Propre du CNRS, Case 907, F-13288 Marseille 09, France

Esko Keski-Vakkuri
Department of Theoretical Physics, University of Helsinki, Siltavuorenpenger 20 C, SF-00170 Helsinki 17, Finland

and

Antti J. Niemi [2]
Research Institute for Theoretical Physics, University of Helsinki, Siltavuorenpenger 20 C, SF-00170 Helsinki 17, Finland

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A geometrical interpretation of path integrals is developed in the space of trajectories. This yields a supersymmetric formulation of a generic path integral, with the supersymmetry resembling the BRST supersymmetry of a first class constrained system. If the classical equation of motion is a Killing vector field in the space of trajectories, the supersymmetry localizes the path integral to classical trajectories and the WKB approximation becomes exact. This can be viewed as a path integral generalization of the Duistermaat-Heckman theorem, which states the conditions for the exactness of the WKB approximation for integrals in a *compact* phase space.

There is an old conjecture [1,2] that also appears in some textbooks [3], and states that in some models corrections to the WKB, or gaussian approximation to a path integral can be ignored. The argument uses Hamilton-Jacobi theory and the formal invariance of phase space path integrals under canonical transformations. The conjecture has been verified in certain cases, but there are also many examples where it fails. Since the WKB approximation is widely used, it is very important to understand when the conjecture is correct and why it can fail.

Independently, the exactness of the WKB approximation has also been investigated in the mathematics literature [4–7]. There, a theorem by Duistermaat and Heckman states that if the exponential of a hamiltonian with torus action is integrated over a *compact* phase space with Liouville measure the integral localizes to the critical points of the hamiltonian, hence the WKB approximation is exact. The derivation of the theorem can be formulated using equivariant cohomology [5], and there are interesting connections to supersymmetry and Witten's work on Morse theory [8]. However, thus far the results have not been extended to path integrals.

In the present Letter we shall investigate the geometrical structure of phase space path integrals. We are particularly interested in the geometry associated with phase space trajectories. From the ensuing structure we conclude that a generic path integral admits a hidden supersymmetry which is very similar to the BRST supersymmetry encountered in the quantization of first class constrained systems. Generalizing techniques of the hamiltonian BRST quantization, we find that if the determinant of Jacobi fields is nontrivial and if the space of trajectories admits a metric tensor for which the classical equation of motion is Killing, this supersymmetry localizes the path integral to classical trajectories. As a consequence the WKB approximation becomes exact, which can be viewed as an extension of the Duistermaat–Heckman theorem to path integrals. Our

[1] Bitnet address: BLAU @ FRCPTM51
[2] Bitnet address: ANIEMI @ FINUHCB

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conditions are satisfied whenever the phase space admits a metric tensor such that the classical hamiltonian determines a Killing vector field. Our results are consistent with recent evaluations of character formulas using phase space path integrals [9], and the observation that for a point particle propagating on a group manifold the classical and quantum Green's functions coincide [10].

As an introduction we shall first present a "false" derivation of the conjectured exactness of the WKB approximation [1,3]. For this we consider the phase space path integral

$$Z = \int [\mathrm{d}p\, \mathrm{d}q] \exp\left(\mathrm{i} \int p\dot{q} - H\right). \tag{1}$$

Hamilton-Jacobi theory tells that whenever $S(q, P; t)$ with P a parameter, solves the Hamilton-Jacobi equation

$$H(q, \partial S/\partial q; t) + \partial S/\partial t = 0, \tag{2}$$

it also generates a (time dependent) canonical transformation between the canonical pairs p_i, q^i and P_i, Q^i such that in the new variables P_i, Q^i the hamiltonian vanishes,

$$p\, \mathrm{d}q - H\, \mathrm{d}t + Q\, \mathrm{d}P = \mathrm{d}S(q, P; t). \tag{3}$$

Eq. (2) is solved by the classical action $S(q_f, P_f; q_i, P_i; t)$ when evaluated along a *classical* trajectory of the system connecting the end points q_i and q_f.

Formally, the measure in (1),

$$[\mathrm{d}p\, \mathrm{d}q] \overset{N\to\infty}{=} \prod_{n=1}^{N} \mathrm{d}p_n \prod_{n=1}^{N-1} \mathrm{d}q_n, \tag{4}$$

differs from the Liouville measure only by an integration over momenta at the end points. Hence, (1) is *formally* equal to

$$Z = \int [\mathrm{d}P\, \mathrm{d}Q] \mathscr{J}(P, Q) \exp\left(-\mathrm{i} \int Q\dot{P}\right) \exp[\mathrm{i}S(q_f, P_f; q_i, P_i; t)], \tag{5}$$

with $\mathscr{J}(P, Q)$ the jacobian. We find that this jacobian is the square root of the Van Vleck determinant, hence

$$Z = \int [\mathrm{d}p\, \mathrm{d}q] \exp\left(\mathrm{i} \int p\dot{q} - H\right) = \sum_{\text{classical}} \sqrt{\det\|\partial^2 S(q_f, q_i)/\partial q_f\, \partial q_i\|}\, \exp[\mathrm{i}S(q_f, q_i)], \tag{6}$$

where the summation extends over all classical trajectories that connect the end points. The Van Vleck determinant can be interpreted as the density of trajectories, and it is the inverse of the determinant of Jacobi fields. Consequently (6) assumes that the determinant of Jacobi fields does not vanish, which means e.g. that there are no coalescing classical trajectories.

The RHS of (6) is equal to the expression that we find, if we apply the WKB approximation to the LHS of (6). Hence (6) suggests that corrections to the WKB approximation can be ignored provided the determinant of Jacobi fields is nontrivial. However, in addition of overlooking difficulties associated with nonlocal, time dependent canonical transformations and the $N \to \infty$ limit in (4), the derivation also ignores e.g. the fact that the kinetic term

$$\int p\dot{q} \sim \sum_{n=1}^{N} p_n(q_n - q_{n-1}) \tag{7}$$

remains invariant only under canonical transformations which act linearly on the fields. As a consequence (6) cannot be justified by a detailed analysis based on a careful discretization of the path integral (1).

For an *integrable* model the previous argument becomes somewhat stronger. For an integrable model there exists a time *independent* canonical transformation to action (I_i) and angle (φ^i) variables such that H becomes a function of I_i only. This has led to a refinement of the conjecture, that at least for certain integrable models the WKB approximation could reliably describe the exact quantum theory [2,3]. However, since the canonical transformation to the action-angle variables is in general highly nonlinear and nonlocal and since it also fails to preserve the polarization of the quantum theory [11], we expect that in general the original and the action-angle path integrals describe inequivalent quantum theories.

In order to systematically analyze the exactness and corrections to the WKB approximation, it is necessary to look for formulations which avoid the difficulties associated with the Hamilton-Jacobi theory and its time dependent and nonlocal canonical transformations. We shall now proceed with such an analysis, which will be based on the geometry of phase space trajectories and the ensuing supersymmetry. For this we consider a general path integral

$$Z=\int [\mathrm{d(Liouville)}]\exp(\mathrm{i}S)=\int [\mathrm{d}\phi^a]\sqrt{\det\|\omega_{ab}\|}\,\exp\left(\mathrm{i}\int \theta_a(\phi)\dot\phi^a-H(\phi)\right), \tag{8}$$

with ϕ^a coordinates in the phase space Γ. The symplectic potential $\theta_a(\phi)$ determines the symplectic two-form by

$$\omega_{ab}(\phi)=\partial_a\theta_b-\partial_b\theta_a \tag{9}$$

and assuming that $\omega_{ab}(\phi)$ is nondegenerate, its matrix inverse defines Poisson brackets by

$$\{A(\phi),B(\phi)\}=(\partial A/\partial\phi^a)\omega^{ab}\,\partial B/\partial\phi^b. \tag{10}$$

In the following we shall find it convenient to introduce a real anticommuting field c^a, so that we can write (8) in the form

$$Z=\int [\mathrm{d}\phi^a\,\mathrm{d}c^a]\exp\left(\mathrm{i}\int \theta_a\dot\phi^a-H(\phi)+\tfrac{1}{2}c^a\omega_{ab}c^b\right). \tag{11}$$

We shall now interpret the classical action S in (8) as an observable on the space of phase space trajectories $\mathrm{T}\Gamma$. The corresponding hamiltonian vector field in $\mathrm{T}\Gamma$ is then identified with the classical equation of motion,

$$\chi^a_S(t)=\int \mathrm{d}t'\,\omega^{ab}\delta(t-t')\frac{\delta}{\delta\phi^b(t')}S(\phi)=\dot\phi^a(t)-\omega^{ab}\,\partial_bH(\phi(t)). \tag{12}$$

Notice that the classical trajectories are zeros of $\chi^a_S(t)$. In order to realize (12) canonically, we introduce a canonical structure on $\mathrm{T}\Gamma$. For this we introduce a variable $\lambda_a(t)$ which is conjugate to the trajectory described by $\phi^a(t)$,

$$\{\lambda_a(t_1),\phi^b(t_2)\}=\delta^b_a\delta(t_1-t_2). \tag{13}$$

We can then identify the components of (12) with

$$\omega_{ab}\chi^b_S=\{\lambda_a,S(\phi)\}=\omega_{ab}\dot\phi^b(t)-\partial_aH(\phi(t)). \tag{14}$$

Similarly, we introduce an anticommuting field $\bar c_a(t)$ which is conjugate to $c^a(t)$,

$$\{\bar c_a(t_1),c^b(t_2)\}=\delta^b_a\delta(t_1-t_2). \tag{15}$$

We can identify $c^a(t)$ with the basis of one-forms $\mathrm{d}\phi^a(t)$ in $\mathrm{T}\Gamma$, and $\bar c_a(t)$ with a basis of vector fields in $\mathrm{T}\Gamma$. Polynomials in $c^a(t)$ then determine the exterior algebra in $\mathrm{T}\Gamma$. The hamiltonian vector field (12) can be represented in the form [#1]

$$\chi_S=\chi^a_S\bar c_a=(\dot\phi^a-\omega^{ab}\,\partial_bH)\bar c_a, \tag{16}$$

[#1] Notice that we use the convention, that a summation over the indices $a, b, \ldots$, also includes an integration over the variable t.

and the exterior differential operator d on TΓ can be defined by the canonical action of

$$\mathrm{d} = \int \mathrm{d}t\, \mathrm{d}\phi^a(t) \frac{\delta}{\delta\phi^a(t)} = \lambda_a c^a. \tag{17}$$

It maps p-forms of TΓ, i.e. p-polynomials in $c^a(t)$ into $(p+1)$-forms.

In the exterior algebra of TΓ we can also introduce an inner multiplication that maps p-forms into $(p-1)$-forms in TΓ. It is defined by the canonical action of

$$\mathrm{i}_S = \chi_S^a \bar{c}_a. \tag{18}$$

Using (17) and (18) we then define the operator

$$Q_S = \mathrm{d} + \mathrm{i}_S. \tag{19}$$

It maps a p-form into a linear combination of a $(p+1)$-form and a $(p-1)$-form. Therefore, if we split the space of exterior forms into its even ($\mathrm{T}\Gamma^+$) and odd ($\mathrm{T}\Gamma^-$) elements, Q_S determines a mapping between $\mathrm{T}\Gamma^+$ and $\mathrm{T}\Gamma^-$. Furthermore, if we interpret the even elements $\mathrm{T}\Gamma^+$ as bosons and the odd elements $\mathrm{T}\Gamma^-$ as fermions, we can interpret Q_S as a supersymmetry operator. The corresponding supersymmetry algebra is

$$\tfrac{1}{2}\{Q_S, Q_S\} = \mathscr{L}_S = c^a\{\lambda_a, \chi_S^b\}\bar{c}_b + \lambda_a \chi_S^a = 2c^a(\delta_a^b\, \partial_t - \partial_a[\omega^{bc}\, \partial_c H])\bar{c}_b + \lambda_a \chi_S^a, \tag{20}$$

where $\mathscr{L}_S$ is the Lie derivative along the vector field χ_S in TΓ. In particular, in the subspace where $\mathscr{L}_S = 0$ the operator Q_S becomes nilpotent and can be viewed as an exterior differential operator. [#2]

The supersymmetry transformation of the canonical variables is

$$\delta_S\phi^a = \{Q_S, \phi^a\} = c^a, \quad \delta_S\lambda_a = \{Q_S, \lambda_a\} = -[\delta_a^b\, \partial_t - \partial_a(\omega^{bc}\, \partial_c H)]\bar{c}_b, \tag{21a, b}$$

$$\delta_S c^a = \{Q_S, c^a\} = \dot{\phi}^a - \omega^{ab}\, \partial_b H, \quad \delta_S \bar{c}_a = \{Q_S, \bar{c}_a\} = \lambda_a. \tag{21c, d}$$

Notice that (21c) vanishes when the classical equation of motion is satisfied, while the vanishing of (21b) determines the Jacobi equation; see ref. [12] for a discussion of similar relations in the supersymmetric formulation in classical mechanics. Furthermore, we find that the action in (11) is supersymmetric,

$$\{Q_S, S + \tfrac{1}{2}c^a \omega_{ab} c^b\} = 0, \tag{22}$$

and the measure in (11) is also (at least formally) supersymmetric.

We shall now assume that TΓ admits a metric tensor $\Omega_{ab}(\phi;\, t, t') = \Omega_{ab}(\phi)\delta(t-t')$ such that the hamiltonian vector field χ_S is a Killing vector,

$$\mathscr{L}_S \Omega = 0. \tag{23}$$

Locally, such a metric tensor can be constructed whenever the original phase space Γ admits a metric tensor for which the hamiltonian vector field of H is a Killing vector. With the help of Ω_{ab} we introduce the dual one-form of the vector field χ_S,

$$\chi_S^* = \Omega_{ab}\chi_S^a c^b = \Omega_{ab}(\dot{\phi}^a - \omega^{ac}\, \partial_c H)c^b, \tag{24}$$

and as a consequence of (23) we find that its Lie derivative along χ_S vanishes,

$$\mathscr{L}_S \chi_S^* = 0. \tag{25}$$

Together with (20) this implies that

$$(\mathrm{d} + \mathrm{i}_S)\chi_S^* = \{Q_S, \chi_S^*\} = \Omega_{ab}\chi_S^a \chi_S^b + c^a(\partial_a \Omega_{bc}\chi_S^b + \Omega_{ab}\, \partial_c \chi_S^b)c^c \tag{26}$$

[#2] Notice that this supersymmetry is quite different from the supersymmetry in the path integral formulation of *classical* mechanics [12]. Here we have generalized the supersymmetry of refs. [5,7] to the loop space, while the supersymmetry in ref. [12] emerges from the Parisi–Sourlas supersymmetry [13].

is supersymmetric, i.e. has a vanishing commutator with Q_S. Notice that (26) is trivially supersymmetric since it is the supersymmetry variation of χ_S^* which is an element in the subspace $\mathscr{L}_S=0$. However, in general the supersymmetry (22) of the action is nontrivial since a generic action $S+\frac{1}{2}c^a\omega_{ab}c^b$ cannot be represented as a supersymmetry variation of some other functional.

Consider the following path integral:

$$Z_\psi=\int [\mathrm{d}\phi^a\,\mathrm{d}c^a]\exp\left(\mathrm{i}\int \theta_a\dot{\phi}^a-H(\phi)+\tfrac{1}{2}c^a\omega_{ab}c^b+\{Q_S,\psi\}\right), \tag{27}$$

where $\psi(\phi, c)$ is a functional such that its Lie derivative along χ_S vanishes,

$$\mathscr{L}_S\psi=0. \tag{28}$$

This means that ψ is in the subspace where Q_S is nilpotent. Formally, the path integral (27) is invariant under the supersymmetry determined by Q_S. For $\psi=0$ (27) reduces to the original path integral (11), and we shall now argue that (27) is *independent* of the functional form of ψ whenever ψ satisfies (28). For this we consider a variation $\psi\to\psi+\delta\psi$ where $\delta\psi$ also satisfies (28),

$$\mathscr{L}_S\delta\psi=0. \tag{29}$$

In order to establish the ψ-independence of (27) it is then sufficient to verify that

$$Z_\psi=Z_{\psi+\delta\psi}. \tag{30}$$

Notice that this is reminiscent of the arguments used in hamiltonian BRST quantization, with Q_S the BRST operator [14]. Indeed, in the subspace where $\mathscr{L}_S=0$ the supersymmetry generator Q_S is nilpotent and can be viewed as a BRST operator. Our statement (30) is then related to the Fradkin-Vilkovisky theorem [14] provided we only consider "gauge fermions" ψ which are in the subspace (28).

In order to establish (30) we consider a *local* supersymmetry transformation parametrized by $\delta\psi$,

$$\phi^a\to\phi^a+\delta\psi\{Q_S,\phi^a\},\quad c^a\to c^a+\delta\psi\{Q_S,c^a\}. \tag{31a, b}$$

The action in (27) is invariant under (31). However, since $\delta\psi$ is a nontrivial function of ϕ^a and c^a the measure fails to be supersymmetric. If we exponentiate (31) and evaluate the pertinent superjacobian, we find that the only effect of (31) is to replace

$$\{Q_S,\psi\}\to\{Q_S,\psi+\delta\psi\} \tag{32}$$

in (27). Provided the boundary conditions in (27) are supersymmetric, we have then verified (30).

We shall now apply (30) to the following family of "gauge fermions":

$$\psi_\xi=\xi\cdot\chi_S^*=\xi\cdot\Omega_{ab}\chi_S^a c^b \tag{33}$$

in (27), with ξ a parameter that scales the metric tensor $\Omega_{ab}\to\Omega_{ab}^\xi=\xi\cdot\Omega_{ab}$. By (25) and linearity of the Lie derivative, these ψ_ξ's satisfy the condition (28). Explicitly, the action in (27) is now

$$S_\xi=\int\{\theta_a\dot{\phi}^a-H+\tfrac{1}{2}c^a\omega_{ab}c^b+\xi\cdot\Omega_{ab}(\dot{\phi}^a-\omega^{ac}\,\partial_c H)(\dot{\phi}^b-\omega^{bd}\,\partial_d H)$$
$$+\xi\cdot c^a[\Omega_{ab}\,\partial_t-\Omega_{bc}\,\partial_a(\omega^{cd}\,\partial_d H)+\partial_a\Omega_{bc}(\dot{\phi}^c-\omega^{cd}\,\partial_d H)]c^b\} \tag{34}$$

and the ψ-independence (30) ensures the $\xi\cdot\Omega_{ab}$-independence of the corresponding path integral (27). In the $\xi\to 0$ limit the path integral reduces to (11), and in the $\xi\to\infty$ limit we get

$$Z_{\xi\to\infty}=\int[\mathrm{d}\phi^a]\sqrt{\det\|\omega_{ab}\|}\,\delta(\chi^a)\sqrt{\det\|\delta\chi^a/\delta\phi_b\|}\,\exp(\mathrm{i}S)$$

$$=\int[\mathrm{d}\phi^a]\sqrt{\det\|\omega_{ab}\|}\,\delta(\dot\phi^a-\omega^{ab}\,\partial_b H)\sqrt{\det\|\delta^a_b\,\partial_t-\partial_b(\omega^{ac}\,\partial_c H)\|}\,\exp\left(\mathrm{i}\int\theta_a\dot\phi^a-H\right)$$

$$=\sum_{\text{classical}}\frac{\sqrt{\det\|\omega_{ab}\|}\,\exp(\mathrm{i}S)}{\sqrt{\det\|\delta^a_b\,\partial_t-\partial_b(\omega^{ac}\,\partial_c H)\|}}.\qquad(35)$$

Notice that the $\xi\cdot\Omega_{ab}$ dependence has indeed disappeared in (35), consistent with the ψ-independence of the path integral (27).

The final result (35) coincides with the WKB approximation of (8). This means that we have established, that if our assumptions are satisfied corrections to the WKB approximation vanish. This result can be viewed as a path integral generalization of the Duistermaat-Heckman theorem. The assumptions that we have introduced are, that the determinant of Jacobi fields in (35) is nontrivial, that the phase space Γ admits a metric tensor for which the hamiltonian vector field of H is a Killing vector, and that the change of variables (31), (32) can be justified. The first two are assumptions on the classical properties of the theory, while the third is an assumption on the properties of the measure in (27).

In the present case the change of variables (31) corresponds to a variation of ξ, which by (33) is a scale transformation of the metric tensor Ω_{ab}. Since (31) is of the same functional form as the changes of variables introduced in the BRST quantization of first class constrained systems [14], we expect that our evaluation of the superjacobian in (32) is as reliable as the corresponding evaluation in BRST quantization. There, corrections to the superjacobian can be related to anomalies and breaking of BRST supersymmetry. Provided our assumptions on the classical properties of the hamiltonian are satisfied, we can then expect that our derivation is correct unless the supersymmetry (20) is broken in the quantum theory by a scale anomaly in $\Omega_{ab}\to\xi\cdot\Omega_{ab}$.

Examples where our conditions are valid include the propagator for a point particle on a group manifold [10], and the evaluation of character formulas using phase space path integrals with the symplectic structure determined by the Kirillov two-form [9]. Further examples will be presented in future publications.

In conclusion, we have established that a *generic* path integral admits a "hidden" supersymmetry, which localizes the path integral to the classical trajectories provided the classical trajectories are sufficiently regular and the supersymmetry remains unbroken in the quantum theory. Our result can be viewed as a path integral generalization of the Duistermaat-Heckman theorem, and it provides a systematic, geometric method for analyzing corrections to the WKB approximation.

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